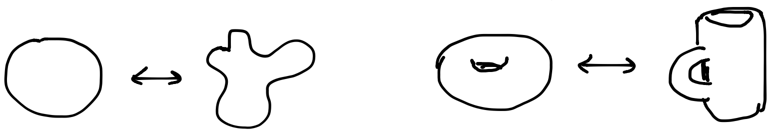
Topology

If X is a set, a topology on X is necessary to study properties that are invariant under continuous deformation.



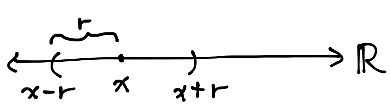
The study of topology came from studying continuity, limits, sequences, etc. on $\mathbb{R}$ (or $\mathbb{R}^n$). All of these definitions can be reformulated in terms of the "topology" of $\mathbb{R}^n$ and they can be generalized to an arbitrary topological space.

Motivating example: Topology of $\mathbb{R}$

Consider $\mathbb{R}$. An "open set" is a subset U of $\mathbb{R}$ where if $x \in U$, then all the elements sufficiently "close" to x are in U as well.

Def: Let $x \in \mathbb{R}$. The <u>open ball</u> of radius r around x is the interval

$$B_r(x) = (x-r, x+r) = \{y \in \mathbb{R} \mid |x-y| < r\}$$



Then a subset of $\mathbb{R}$ is open if each point has an open ball around it. More precisely:

Def: $U \subseteq \mathbb{R}$ is <u>open</u> if $\forall\, x \in U \ \exists\ r > 0$ s.t. $B_r(x) \subseteq U$

Ex: $(0,1]$ is <u>not</u> open. Every open ball $B_r(1)$ contains elements greater than 1. Specifically $1+\frac{r}{2} \in B_r(1)$, but $1+\frac{r}{2} \notin (0,1]$.

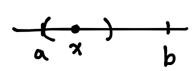
EX: If $X \subseteq \mathbb{R}$ is a finite set of points, then X is not open since for $x \in X$, $B_r(x)$ is infinite $\forall\ r>0$. Thus $B_r(x) \nsubseteq X$.

EX: $\emptyset$ and $\mathbb{R}$ are both open. Do you see why?

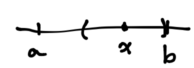
Theorem: If $a,b \in \mathbb{R}$ w/ $a<b$ then the open interval $(a,b) \subseteq \mathbb{R}$ is open.

Proof: Let $x \in (a,b)$. Set $r = \min\{x-a,\ b-x\}$.

Then $B_r(x) = (x-r,\ x+r)$



or



Then $r \leq x-a$, so

$$a \leq x-r$$

and $r \leq b-x$, so $x+r \leq b$.

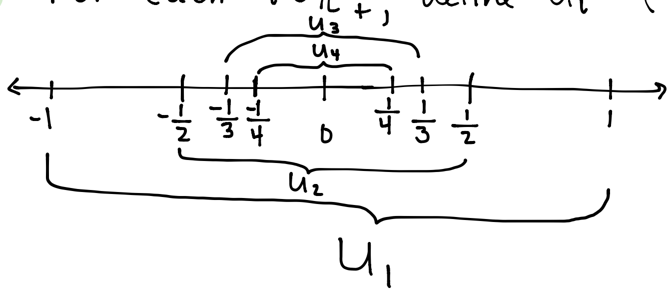
Thus $x-a \leq x-r \leq x+r \leq x+a$, so $B_r(x) \subseteq (a,b)$. $\square$

Thm: If $\mathcal{C}$ is a (possibly infinite!) collection of open sets in $\mathbb{R}$, then $V = \bigcup_{U \in \mathcal{C}} U$ is open.

Proof: Let $x \in V$. Then $x \in U$ for some $U \in \mathcal{C}$.
Since U is open, $\exists\ r>0$ s.t. $B_r(x) \subseteq U$.
Thus, $B_r(x) \subseteq V$, so V is open.

The above statement <u>doesn't</u> hold for intersections:

Ex: For each $i \in \mathbb{Z}_+$, define $U_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$



Then U_i is open for each i.

Set $V = \bigcap_{i \in \mathbb{Z}_+} U_i$.

Then $0 \in U_i \ \forall \ i$, so $0 \in V$. However $\forall \ r > 0$, $\exists \ i \in \mathbb{Z}_+$ s.t. $\frac{1}{i} < r$. (Why? We can find $i > \frac{1}{r}$, so $\frac{1}{i} < r$.)

Thus $B_r(x) \not\subseteq U_i$, so $B_r(x) \not\subseteq V$.

However, if we just take a finite intersection, then the statement holds:

Theorem: If $\{U_1, \dots, U_n\}$ is a finite collection of open sets, then

$$V = \bigcap_{i \in \{1,\dots,n\}} U_i \text{ is open.}$$

Pf: Let $x \in V$. Then $x \in U_i \ \forall i$, so for each i, $\exists \ r_i > 0$ s.t.

$$B_{r_i}(x) \subseteq U_i.$$

Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $B_r(x) \subseteq B_{r_i}(x) \ \forall \ i$, so

$B_r(x) \subseteq U_i \ \forall i$, so $B_r(x) \subseteq V$, as desired. $\square$